

GLOBAL RIGIDITY OF GENERIC FRAMEWORKS ON CONCENTRIC CYLINDERS

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ABSTRACT. We show that a generic framework (G, p) in \mathbb{R}^3 whose vertices are constrained to lie on a family of concentric cylinders is globally rigid if and only if G is a complete graph on at most four vertices or G is both redundantly rigid and 2-connected.

1. INTRODUCTION

A (bar-joint) framework (G, p) in \mathbb{R}^d is the combination of a finite, simple graph $G = (V, E)$ and a realisation $p : V \rightarrow \mathbb{R}^d$. The framework (G, p) is rigid if every edge-length preserving continuous motion of the vertices arises as a congruence of \mathbb{R}^d . Moreover (G, p) is globally rigid if every framework (G, q) with the same edge lengths as (G, p) arises from a congruence of \mathbb{R}^d .

In general it is NP-hard to determine the rigidity or global rigidity of a given framework [1, 19]. These problems become more tractable, however, for generic frameworks. It is known that both the rigidity and global rigidity of a generic framework (G, p) in \mathbb{R}^d depend only on the underlying graph G , see [2, 7]. We say that G is *rigid* or *globally rigid in \mathbb{R}^d* if some/every generic realisation of G in \mathbb{R}^d has the corresponding property. Combinatorial characterisations of generic rigidity and global rigidity in \mathbb{R}^d have been obtained when $d \leq 2$, see [13, 9], and these characterisations give rise to efficient combinatorial algorithms to decide if these properties hold. In higher dimensions, however, no combinatorial characterisations or algorithms are yet known.

We consider the situation where (G, p) is a framework in \mathbb{R}^3 whose vertices are constrained to lie on a fixed surface. Combinatorial characterisations for generic rigidity in this context were established for surfaces consisting of families of concentric spheres [21, 17], cylinders [17], and cones [18]. In particular it was shown that a generic realisation of a graph G on a family of concentric spheres is rigid if and only if G is rigid in the plane.

The characterisation of rigidity for a generic framework (G, p) on a family of cylinders uses the *simple (2, 2)-sparse matroid* for G . This is the matroid $\mathcal{M}_{2,2}^*(G)$ on $E(G)$ in which a set of edges F is *independent* if and only if $|F'| \leq 2|V(F')| - 2$ for all $\emptyset \neq F' \subseteq F$, with strict inequality when $|F'| = 2$.

Theorem 1.1. *Let (G, p) be a generic framework on a family of concentric cylinders. Then (G, p) is rigid if and only if G is a complete graph on at most 3 vertices or $\mathcal{M}_{2,2}^*(G)$ has rank $2|V(G)| - 2$.*

This characterisation implies that the rigidity of a generic framework (G, p) on a family of concentric cylinders depends only on the underlying graph G , and is the same for *all* generic realisations of G on *all* families of concentric cylinders. We will say that G is *rigid*

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on the cylinder if it is generically rigid on some/all families of concentric cylinders and that G is *redundantly rigid on the cylinder* if $G - e$ is rigid on the cylinder for all edges e of G .

We next consider global rigidity on surfaces. One possible motivation for doing this arises in wireless sensor network localisation problems, see [15, Page 2] and the references therein. Connelly and Whiteley [5] showed that a graph G is generically globally rigid on the sphere if and only if it is generically globally rigid in the plane (which holds if and only if G is 3-connected and redundantly rigid in the plane by [9]). In [11], necessary combinatorial conditions were established for a framework on a surface to be generically globally rigid. The conditions, redundant rigidity and k -connectivity (where the integer k depends on the chosen surface), are analogous to those which characterise generic global rigidity on the plane and the sphere. These conditions were conjectured to also be sufficient for families of cylinders and cones. In this paper we verify this conjecture for generic families of concentric cylinders.

Theorem 1.2. *Let (G, p) be a generic framework in \mathbb{R}^3 . Then (G, p) is globally rigid on the family of concentric cylinders induced by p if and only if G is either a complete graph on at most four vertices or G is 2-connected and redundantly rigid on the cylinder.*

The key step in proving sufficiency in Theorem 1.2 is the following result which deals with the special case when G is 2-connected and redundantly rigid with the minimum possible number of edges. Theorem 1.1 implies that $|E| \geq 2|V| - 1$ whenever $G = (V, E)$ is redundantly rigid, and that equality holds only if E is a circuit in $\mathcal{M}_{2,2}^*(G)$. We will abuse terminology and say that G is a circuit in $\mathcal{M}_{2,2}^*$ whenever this occurs.

Theorem 1.3. *Let G be a circuit in $\mathcal{M}_{2,2}^*$ and (G, p) be a generic framework in \mathbb{R}^3 . Then (G, p) is globally rigid on the family of concentric cylinders induced by p .*

Note that the genericity condition of Theorems 1.2 and 1.3 is stronger than in Theorem 1.1 since it requires the cylinders in the family to have generic radii.

We will need the following three results to prove Theorem 1.3. The first is a decomposition result for circuits which uses the graph operations defined in Figure 1.

Theorem 1.4. [16, Lemmas 3.1, 3.2, 3.3] *Suppose G_0, G_1, G_2 are graphs with $|E(G_i)| = 2|V(G_i)| - 2$ for all $0 \leq i \leq 2$ and that G_0 is an j -join of G_1 and G_2 for some $1 \leq j \leq 3$. Then G_0 is a circuit in $\mathcal{M}_{2,2}^*$ if and only if both G_1 and G_2 are circuits in $\mathcal{M}_{2,2}^*$.*

The second result we shall need is a recursive construction for circuits which uses the i -join operations as well as the 1-extension operation which deletes an edge xy from a graph G and then adds a new vertex v and three new edges vx, vy, vz for some vertex $z \neq x, y$. The recursion begins with the three circuits defined in Figure 2.

Theorem 1.5. [16, Theorem 1.1] *Suppose G is a circuit in $\mathcal{M}_{2,2}^*$. Then G can be obtained from either $K_5 - e$, H_1 or H_2 by recursively applying the operations of 1-extension, and 1-, 2- and 3-join.*

The third result gives an algebraic sufficient condition for global rigidity on a generic family of cylinders in terms of stress matrices (which will be defined in Section 3).

Theorem 1.6. [12, Theorem 6.2] *Let (G, p) be a generic framework in \mathbb{R}^3 with $n \geq 3$ vertices. Suppose that (G, p) has an equilibrium stress for which the associated stress matrix has rank $3n - 6$. Then (G, p) is globally rigid on the family of concentric cylinders induced by p .*

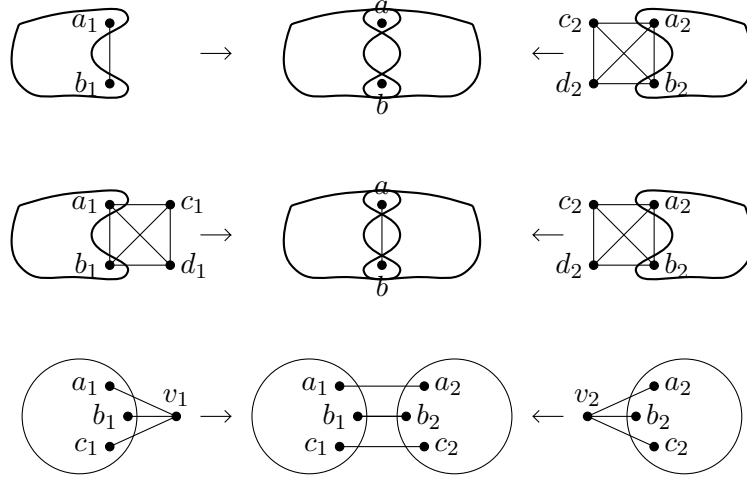


FIGURE 1. The 1-, 2- and 3-join operations. The 1- and 2-join operations form the graphs in the centre by merging a_1 and a_2 into a , and b_1 and b_2 into b . We write $G = G_1 *_i G_2$ to mean G is an i -join of G_1 and G_2 .

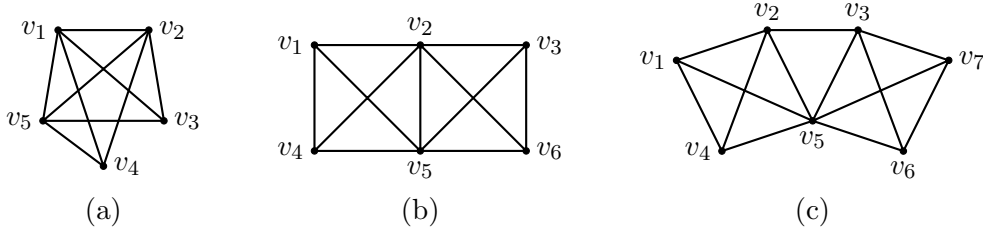


FIGURE 2. The graphs $K_5 - e$, H_1 and H_2 .

In Section 2 we will give a new recursive construction for circuits in $\mathcal{M}_{2,2}^*$. In Section 3 we give formal definitions for infinitesimal rigidity and equilibrium stresses for frameworks on concentric cylinders. In Section 4 we will show that each of the operations in the recursive construction preserves the property of having a maximum rank stress matrix. We will apply these results to show that every circuit in $\mathcal{M}_{2,2}^*$ is generically globally rigid and then use this to completely characterise generic global rigidity in Section 5.

2. RECURSIVE CONSTRUCTION

We will refine the recursive construction for circuits given in Theorem 1.5. The aim is to make the steps in the recursion as simple as possible since we have to show they preserve global rigidity, or, more precisely, preserve the property of having a maximum rank stress matrix. To this end we introduce two ‘new’ operations. Given a circuit $G = (V, E)$, the first operation, K_4^- -extension, is just a 1-join of G and H_1 . The second operation, *generalised vertex split*, is defined as follows: choose $v \in V$ and a partition N_1, N_2 of the neighbours of v ; then delete v from G and add two new vertices v_1, v_2 joined to N_1, N_2 , respectively; finally add two new edges v_1v_2, v_1x for some $x \in V \setminus N_2$. These operations are illustrated in Figures 3 and 4.

The usual vertex splitting operation, see [22], is the special case when x is chosen to be a neighbour of v_2 . Note also that the special case when v_1 has degree 3 (and $v_2 = v$) is the

1-extension operation. At times it will be convenient to work directly with the 1-extension operation itself, for example in Theorem 2.2 below.

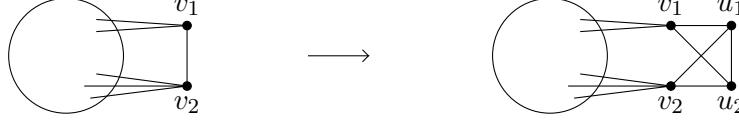


FIGURE 3. K_4^- -extension.

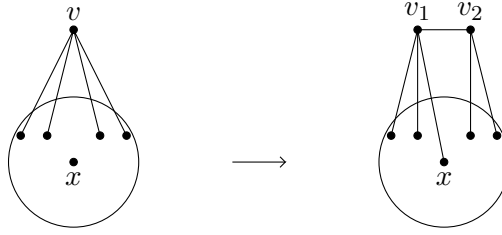


FIGURE 4. Generalised vertex split.

Our simplified recursive construction is the following:

Theorem 2.1. *Suppose G is a circuit in $\mathcal{M}_{2,2}^*$. Then G can be obtained from either $K_5 - e$ or H_1 by recursively applying the operations of K_4^- -extension and generalised vertex split, in such a way that each of the intermediate graphs is a circuit.*

Since the K_4^- -extension operation is a special case of the 1-join operation, it must necessarily preserve the property of being a circuit by Theorem 1.4. The 1-extension operation also preserves the property of being a circuit by [16, Lemma 2.1], but the generalised vertex split operation may not.

We will refer to the inverse operations to those used in Theorem 2.1 as K_4^- -reduction and edge-reduction, respectively, and to the inverse of the 1-extension operation as 1-reduction. We say that an application of each of these operations is *admissible* if, when we apply it to a circuit, we obtain a smaller circuit. Theorem 1.4 implies that K_4^- -reduction will always be admissible but the operations of 1-reduction and, more generally, edge-reduction may not.

We say that a vertex is a node of a graph if it has degree three, and that a node v in a circuit G is *admissible* if we can construct a smaller circuit from G by applying a 1-reduction operation at v . It was shown in [16, Theorem 1.2] that every 3-connected, essentially 4-edge-connected circuit other than $K_5 - e$ has at least two admissible nodes. We first extend this result by allowing ‘trivial’ 2-vertex-cuts. We need the following definitions.

A *2-vertex-separation* of a graph G is a pair of induced subgraphs $F_1 = (V_1, E_1)$ and $F_2 = (V_2, E_2)$ such that $F_1 \cup F_2 = G$, $|V_1 \cap V_2| = 2$ and $V_1 \setminus V_2 \neq \emptyset \neq V_2 \setminus V_1$. The 2-separation (F_1, F_2) is *nontrivial* if $F_i \neq K_4$ for each $i = 1, 2$. A *3-edge-separation* is a pair of vertex-disjoint induced subgraphs (F_1, F_2) such that $F_1 \cup F_2 = G - S$ for some set $S \subseteq E$ with $|S| = 3$. It is *nontrivial* if S is a set of three independent edges in G . An

atom of G is a subgraph F such that F is an element of a nontrivial 2-vertex separation or 3-edge-separation of G and no proper subgraph of F has this property.

We also need to extend the concepts of circuits and admissible nodes to multigraphs. We define the $(2, 2)$ -sparse matroid $\mathcal{M}_{2,2}(H)$ on the edge set of a multigraph H by defining a set of edges F of H to be independent if $|F'| \leq 2|V(F')| - 2$ for all $\emptyset \neq F' \subseteq F$. Suppose that H is a circuit in $\mathcal{M}_{2,2}$. We say that a node v of H is *allowable* if applying the 1-reduction operation at v produces a smaller circuit in $\mathcal{M}_{2,2}$ and does not create any new multiple edges.

Theorem 2.2. *Suppose G is a circuit in $\mathcal{M}_{2,2}^*$ which is distinct from $K_5 - e$, H_1 and H_2 , and that G has no nontrivial 2-vertex separation and no nontrivial 3-edge-separation. Then G has at least two admissible nodes.*

Proof. If G is 3-connected then the statement is [16, Theorem 1.2] so we may assume that G is not 3-connected. Since G is a circuit, G is 2-connected by [16, Lemma 2.3]. Since G has no nontrivial 2-vertex separation and $G \neq H_1$, every 2-vertex-separation (F_i, F_j) of G has $F_i = K_4$ and $F_j \neq K_4$. For every such 2-vertex-separation with $V(F_i) \cap V(F_j) = \{x_i, y_i\}$, we consider the multigraph H formed by deleting $V(F_i) - \{x_i, y_i\}$ and $E(F_i)$ from G , and adding two copies of the edge $x_i y_i$. Since $G \neq H_2$, H is a 3-connected circuit in $\mathcal{M}_{2,2}$ with no nontrivial 3-edge-separation. Also, each x_i and y_i has degree at least 4 in H ; otherwise there would be a nontrivial 2-vertex-separation in G . It follows that every allowable node in H is an admissible node of G . We can now show that H contains two allowable nodes using the proof technique of [16, Theorem 1.2], see [16, Section 4]. \square

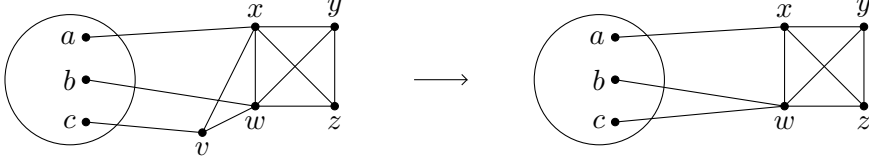
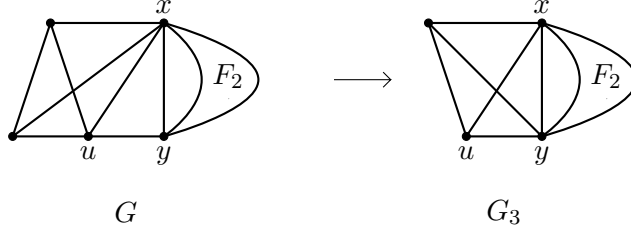
Theorem 2.1 will follow immediately from the following reduction result.

Theorem 2.3. *Let $G = (V, E)$ be a circuit in $\mathcal{M}_{2,2}^*$ distinct from $K_5 - e$ and H_1 . Then G has either a K_4^- -reduction or an admissible edge-reduction.*

Proof. We proceed by induction on $|V|$. If G has no nontrivial 2-separation or 3-edge-separation then either $G = H_2$ and can be reduced to H_1 by an admissible edge-reduction operation, or else G has two admissible nodes by Theorem 2.2, and hence has two admissible edge-reductions. Thus we may suppose that this is not the case. It follows that G has at least two distinct atoms. Let F_1 be an atom of G and (F_1, F_2) be the nontrivial separation which contains F_1 . Consider the following cases.

Case 1: (F_1, F_2) is a nontrivial 3-edge-separation. Let G_i be obtained from G by contracting F_{3-i} to a single vertex z_i for each $i \in \{1, 2\}$. Then $G = G_1 *_3 G_2$ so G_1, G_2 are circuits by Theorem 1.4. Since F_1 is an atom, G_1 has no nontrivial 2-vertex-separation or 3-edge-separation. If $G_1 \notin \{K_5 - e, H_1, H_2\}$ then G_1 has an admissible node v distinct from z_1 by Theorem 2.2, and v will be an admissible node in G by Theorem 1.4.

If $G_1 = K_5 - e$ then G is as shown in Figure 5. We can apply an edge-reduction operation which deletes the edge xw and then contracts the edge xa to construct the graph G'_2 in Figure 5. Then G'_2 is a circuit since it can be obtained from G_2 by two 1-extensions. Hence this edge-reduction operation is admissible. If $G_1 = H_2$ then we will contradict the assumption that F_1 is an atom of G . It remains to consider the subcase when $G = H_1$, which is illustrated in Figure 6. In this case the vertex v is admissible, since performing a 1-reduction on v gives the graph G_3 on the right of Figure 6, and we have $G_3 = G_2 *_1 H_2$ so G_3 is a circuit by Theorem 1.4.

FIGURE 5. The subcase $G_1 = K_5 - e$ in Case 1.FIGURE 6. The subcase $G_1 = H_1$ of Case 1.FIGURE 7. The subcase $G_1 = H_2$ in Case 2.

Case 2: (F_1, F_2) is a nontrivial 2-vertex-separation, $V(F_1) \cap V(F_2) = \{x, y\}$ and $xy \in E$. Let G_i be obtained from F_i by adding two new vertices $\{w, z\}$ and five new edges $\{wx, wy, wz, xz, yz\}$. Then $G = G_1 *_2 G_2$ so G_1 and G_2 are circuits by Theorem 1.4. Since F_1 is an atom, G_1 has no nontrivial 2-vertex-separation or 3-edge-separation. We have $G_1 \neq K_5 - e$ since G_1 is not 3-connected and $G_1 \neq H_1$ since the 2-separation (F_1, F_2) is nontrivial.

Suppose $G_1 = H_2$. Then G is as shown in Figure 7. Let G_3 be obtained from G by the edge-reduction which deletes xy and then contracts uy . This is an admissible edge-reduction since $G = H_2 *_2 G_3$ so G_3 is a circuit by Theorem 1.4.

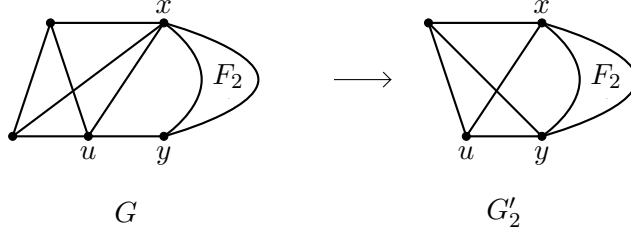
Hence we may suppose that $G_1 \notin \{K_5 - e, H_1, H_2\}$. Then G_1 has an admissible node v by Theorem 2.2. The vertex v will be distinct from w, z since they are not admissible, and distinct from x, y since they are not nodes in G_1 . Thus v will be an admissible node in G .

Case 3: (F_1, F_2) is a nontrivial 2-vertex-separation, $V(F_1) \cap V(F_2) = \{x, y\}$ and $xy \notin E$. We have

$$|E(F_1)| + |E(F_2)| = |E(G)| = 2|V(G)| - 1 = 2|V(F_1)| + 2|V(F_2)| - 5$$

and $|E(F_i)| \leq 2|V(F_i)| - 2$ for each $i = 1, 2$ so $2|V(F_i)| - 3 \leq |E(F_i)| \leq 2|V(F_i)| - 2$. Consider the following subcases.

Subcase 3.1: $|E(F_1)| = 2|V(F_1)| - 3$. Let G_1 be obtained from F_1 by adding two new vertices $\{w, z\}$ and six new edges $\{wx, wy, wz, xy, xz, yz\}$, and $G_2 = F_2 + xy$. Then $G =$

FIGURE 8. The subcase $G_1 = H_2$ in Case 3.1.

$G_1 *_1 G_2$ so G_1 and G_2 are circuits by Theorem 1.4. Since F_1 is an atom, G_1 has no nontrivial 2-vertex-separation or 3-edge-separation. We have $G_1 \neq K_5 - e$ since G_1 is not 3-connected. If $G_1 = H_1$ then G_2 is a K_4^- -reduction of G . Hence we may assume that $G_1 \neq H_1$.

Suppose $G_1 = H_2$. Then G is as shown in Figure 8. Let G'_2 be obtained from G by the edge-reduction which deletes xu and then contracts uy . This is an admissible edge-reduction since $G'_2 = H_1 *_1 G_2$ so G'_2 is a circuit by Theorem 1.4.

Hence we may suppose that $G_1 \notin \{K_5 - e, H_1, H_2\}$. Then G_1 has an admissible node v by Theorem 2.2. The vertex v will be distinct from w, z since they are not admissible, and distinct from x, y since they are not nodes in G_1 . Thus v will be an admissible node in G .

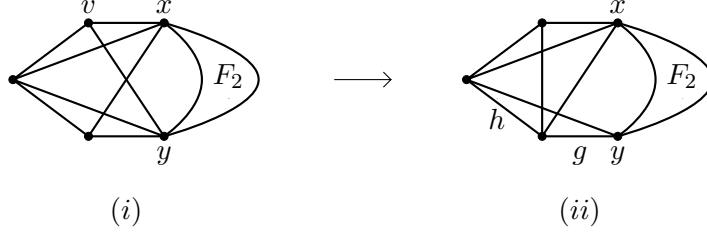
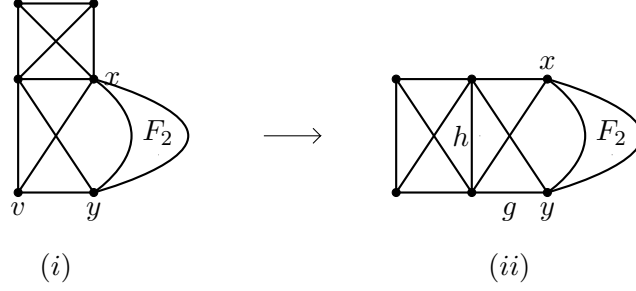
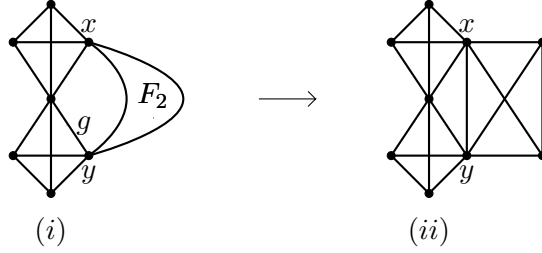
Subcase 3.2: $|E(F_1)| = 2|V(F_1)| - 2$. Let G_2 be obtained from F_2 by adding two new vertices $\{w, z\}$ and six new edges $\{wx, wy, wz, xy, xz, yz\}$, and $G_1 = F_1 + xy$. Then $G = G_1 *_1 G_2$ so G_1 and G_2 are circuits by Theorem 1.4. We may apply the argument used in Subcase 3.1 (to show that $G_1 \notin \{K_5 - e, H_1, H_2\}$) to deduce that $G_2 \notin \{K_5 - e, H_1, H_2\}$. By induction, G_2 has either a K_4^- -reduction or an admissible edge-reduction. If G_2 has a K_4^- -reduction, then the same reduction will exist in G . Hence we may suppose that G_2 has an admissible edge-reduction which deletes an edge e then contracts an adjacent edge f . If $e, f \in E(F_2)$, then the same edge-reduction will be admissible in G , so we may assume that this is not the case. Since the edge-reduction is admissible in G_2 we must have $f \notin \{wx, wy, wz, xy, xz, yz\}$, $e = xy$, and f is incident to either x or y . Without loss of generality we may suppose that f is incident with y . Let G'_2 be the circuit created by this edge-reduction and let G''_2 be the graph obtained from G'_2 by deleting the vertices w, z and adding the edge xy . Since $G'_2 = G''_2 *_1 H_1$, G''_2 is a circuit by Theorem 1.4.

We next consider G_1 . Since F_1 is an atom, G_1 has no nontrivial 2-vertex-separation or 3-edge-separation.

Suppose $G_1 = K_5 - e$. Then G has the structure of one of the graphs shown in Figure 9. If case (i) occurs then the 1-reduction of G which deletes v and adds the edge xy gives the circuit G_2 , so is admissible. If case (ii) occurs then the edge-reduction of G which deletes h and contracts g also gives the circuit G_2 , so is admissible.

Suppose $G_1 = H_1$. Since F_1 is an atom, G would have the structure of one of the graphs shown in Figure 10. If case (i) occurs then the 1-reduction of G which deletes v and adds the edge xy gives the circuit $G_2 *_2 H_2$, so is admissible. If case (ii) occurs then the edge-reduction of G which deletes h and contracts g also gives the circuit $G_2 *_2 H_2$, so is admissible.

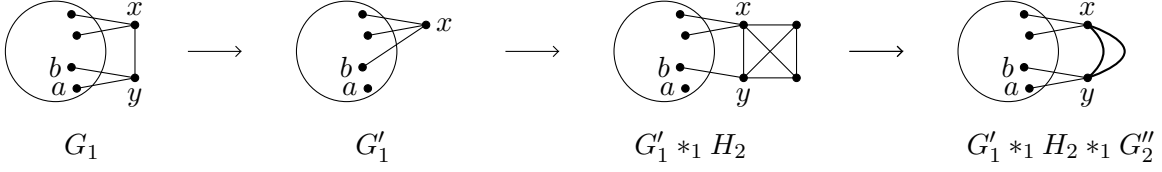
Suppose $G_1 = H_2$. Since F_1 is an atom, G would have the structure of the graph shown in Figure 11(i). We can now use the admissible edge-reduction of G_2 to G'_2 described in the first paragraph of this subcase. Consider the edge-reduction of G which deletes the edge g shown in Figure 11(i) and contracts the edge $e \in E(F_2)$. This is admissible since it gives

FIGURE 9. The subcase $G_1 = K_5 - e$ of Case 3.2.FIGURE 10. The subcase $G_1 = H_1$ of Case 3.2.FIGURE 11. The subcase $G_1 = H_2$ of Case 3.2.

the circuit $G'_2 *_2 H_3$, where H_3 is the circuit shown in Figure 11(ii). (Note that H_3 is a circuit since $H_3 = H_1 *_1 H_2$.)

Hence we may assume that $G_1 \notin \{K_5 - e, H_1, H_2\}$. Then G_1 has two admissible nodes by Theorem 2.2. It can be seen that any admissible node of G_1 which is distinct from x, y is an admissible node of G , so we may suppose that x and y are the only admissible nodes of G_1 . Then both x and y have degree three in G_1 . Let a, b, x be the neighbours of y in G_1 . Since x has degree two in $G_1 - y$, the admissible 1-reduction at y constructs a new graph G'_1 from G_1 by deleting y and then adding a new edge from x to either a or b , say b . We use the admissible reduction of G_2 to G''_2 described in the first paragraph of this subcase. Let G' be the graph obtained from G by performing an edge-reduction which deletes the edge ay and then contracts the edge $f \in E(F_2)$. Then $G' = G'_1 *_1 H_2 *_1 G''_2$ and hence G' is a circuit, see Figure 12. Thus the edge-reduction which transforms G to G' is admissible. \square

To see that the K_4^- -extension operation is needed in Theorem 2.3, observe that we can construct graphs which do not admit admissible edge-reductions as follows. Take any circuit H in $\mathcal{M}_{2,2}^*$, and apply the K_4^- -extension to every single edge of G . The resulting graph G has two types of edges. Those edges with no end-vertices in H are contained in two triangles

FIGURE 12. The subcase of Case 3.2 when $G_1 \notin \{K_5 - e, H_1, H_2\}$.

so any edge-reduction which contracts such an edge results in a non-simple graph. The remaining edges, those with exactly one end-vertex in H , are not admissible either since any edge-reduction which contracts such an edge results in a graph containing a vertex of degree two.

3. RIGIDITY AND STRESS MATRICES

Let $G = (V, E)$ be a graph with $V = \{v_1, \dots, v_n\}$. We will consider realisations of G on an ordered family of (not necessarily distinct) concentric cylinders $\mathcal{Y} = (\mathcal{Y}_1, \mathcal{Y}_2, \dots, \cup \mathcal{Y}_n)$ where $\mathcal{Y}_i = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r_i\}$ and $r = (r_1, \dots, r_n)$ is a vector of positive real numbers. A *framework* (G, p) on \mathcal{Y} is an ordered pair consisting of a graph G and a realisation p such that $p(v_i) \in \mathcal{Y}_i$ for all $v_i \in V$.

Two frameworks (G, p) and (G, q) on \mathcal{Y} are *equivalent* if $\|p(v_i) - p(v_j)\| = \|q(v_i) - q(v_j)\|$ for all edges $v_i v_j \in E$. Moreover (G, p) and (G, q) are *congruent* if $\|p(v_i) - p(v_j)\| = \|q(v_i) - q(v_j)\|$ for all pairs of vertices $v_i, v_j \in V$. The framework (G, p) is *globally rigid* on \mathcal{Y} if every equivalent framework (G, q) on \mathcal{Y} is congruent to (G, p) . It is *rigid* on \mathcal{Y} if there exists an $\epsilon > 0$ such that every framework (G, q) on \mathcal{Y} which is equivalent to (G, p) , and has $\|p(v_i) - q(v_i)\| < \epsilon$ for all $1 \leq i \leq n$, is congruent to (G, p) . It is *generic* on \mathcal{Y} if $\text{td}[\mathbb{Q}(r, p) : \mathbb{Q}(r)] = 2n$.

An *infinitesimal flex* s of (G, p) on \mathcal{Y} is a map $s : V \rightarrow \mathbb{R}^3$ such that $s(v_i)$ is tangential to \mathcal{Y}_i at $p(v_i)$ for all $v_i \in V$ and $(p(v_j) - p(v_i)) \cdot (s(v_j) - s(v_i)) = 0$ for all $v_j v_i \in E$. The framework (G, p) is *infinitesimally rigid* on \mathcal{Y} if every infinitesimal flex is an infinitesimal isometry of \mathbb{R}^3 . It was shown in [17] that a generic framework (G, p) on any family of concentric cylinders is rigid if and only if it is a complete graph on at most three vertices or is infinitesimally rigid.

The *rigidity matrix* $R_{\text{cyl}}(G, p)$ is the $(|E| + |V|) \times 3|V|$ matrix

$$R_{\text{cyl}}(G, p) = \begin{pmatrix} R(G, p) \\ S(G, p) \end{pmatrix}$$

where: $R(G, p)$ has rows indexed by E and 3-tuples of columns indexed by V in which, for $e = v_i v_j \in E$, the submatrices in row e and columns v_i and v_j are $p(v_i) - p(v_j)$ and $p(v_j) - p(v_i)$, respectively, and all other entries are zero; $S(G, p)$ has rows indexed by V and 3-tuples of columns indexed by V in which, for $v_i \in V$, the submatrix in row v_i and column v_i is $\bar{p}(v_i) = (x_i, y_i, 0)$ when $p(v_i) = (x_i, y_i, z_i)$.

An *equilibrium stress* for a framework (G, p) on \mathcal{Y} is a pair (ω, λ) , where $\omega : E \rightarrow \mathbb{R}$ and $\lambda : V \rightarrow \mathbb{R}$ and (ω, λ) belongs to the cokernel of $R_{\text{cyl}}(G, p)$. Thus (ω, λ) is an equilibrium stress for (G, p) on \mathcal{Y} if and only if

$$(3.1) \quad \sum_{j=1}^n \omega_{ij} (p(v_i) - p(v_j)) + \lambda_i \bar{p}(v_i) = 0 \text{ for all } 1 \leq i \leq n,$$

where ω_{ij} is taken to be equal to ω_e if $e = v_i v_j \in E$ and to be equal to 0 if $v_i v_j \notin E$.

Given a stress (ω, λ) for a framework (G, p) on \mathcal{Y} we define $\Omega = \Omega(\omega)$ to be the $n \times n$ symmetric matrix with off-diagonal entries $-\omega_{ij}$ and diagonal entries $\sum_j \omega_{ij}$, and $\Lambda = \Lambda(\lambda)$ to be the $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. The *stress matrix* associated to (ω, λ) is the $3n \times 3n$ symmetric matrix

$$\Omega_{\text{cyl}}(\omega, \lambda) = \begin{bmatrix} \Omega + \Lambda & 0 & 0 \\ 0 & \Omega + \Lambda & 0 \\ 0 & 0 & \Omega \end{bmatrix}.$$

We show in [12] that $\text{rank } \Omega_{\text{cyl}}(\omega, \lambda) \leq 3n - 6$. We will say that $\Omega_{\text{cyl}}(\omega, \lambda)$ *has maximum rank* when equality occurs.

4. K_4^- -EXTENSIONS AND GENERALISED VERTEX SPLITTING

Theorem 1.6 tells us that if (G, p) is generic in \mathbb{R}^3 and $\Omega_{\text{cyl}}(\omega, \lambda)$ is a maximum rank stress matrix for (G, p) , then (G, p) is globally rigid on \mathcal{Y}^p , where \mathcal{Y}^p is the unique family of concentric cylinders induced by p . We will prove Theorem 1.3 by showing that both of the recursive operations used in Theorem 2.1 preserve the property of having a maximum rank stress matrix.

We will need the the following elementary tool from linear algebra. Suppose $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a block matrix and A is invertible. Then the *Schur complement* of A in M is the matrix $F = D - CA^{-1}B$ and we have

$$(4.1) \quad \text{rank } M = \text{rank } A + \text{rank } F.$$

We will also need the following lemma.

Lemma 4.1. [12, Lemmas 7.2 and 7.3] *Suppose (G, p) is an infinitesimally rigid framework on a family of concentric cylinders \mathcal{Y} . Then (G, q) is infinitesimally rigid on \mathcal{Y}^q for all generic $q \in \mathbb{R}^{3n}$. Moreover, if (ω, λ) is an equilibrium stress for (G, p) on \mathcal{Y} with $\text{rank } \Omega_{\text{cyl}}(\omega, \lambda) = 3n - 6$, then (G, q) has an equilibrium stress (ω', λ') on \mathcal{Y}^q with $\text{rank } \Omega_{\text{cyl}}(\omega', \lambda') = 3n - 6$ for all generic $q \in \mathbb{R}^{3n}$.*

We first consider the K_4^- -extension move.

Theorem 4.2. *Suppose $G = (V, E)$ and $G_1 = (V_1, E_1)$, with $v_2v_3 \in E_1$, are chosen so that $G_1 - v_2v_3$ is rigid and G is a K_4^- -extension of G_1 . Let (G, p) be a generic realisation of G in \mathbb{R}^3 and let p_1 be the restriction of p to G_1 . Suppose (G_1, p_1) has a maximum rank stress matrix on \mathcal{Y}^{p_1} . Then (G, p) has a maximum rank stress matrix on \mathcal{Y}^p .*

Proof. Let $V_1 = \{v_2, v_3, \dots, v_n\}$ and suppose that G is constructed from G_1 by deleting v_2v_3 , adding two new vertices v_0, v_1 and five new edges $v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_1v_3$. We may use the isometries of \mathcal{Y}^{p_1} to move (G, p_1) so that $p_1(v_2) = (0, 1, 0)$. Let $p_1(v_i) = (x_i, y_i, z_i)$ for $3 \leq i \leq n$. Define $q : V \rightarrow \mathbb{R}^3$ by putting $q(v) = p_1(v)$ for all $v \in V_1$ and choosing $q(v_0) = (0, 1, 1)$ and $q(v_1) = (1, y_3, z_3)$. (We choose these values for q so that the rows of $R_{\text{cyl}}(G + v_2v_3, q)$ labelled by the vertices and edges of the subgraph H of $G + v_2v_3$ induced by $\{v_0, v_1, v_2, v_3\}$, are dependent. This will enable us to construct an equilibrium stress for (G, q) by combining equilibrium stresses for (G_1, p_1) and $(H, q|_H)$ in such a way that the net stress on v_2v_3 is zero.)

We first show that (G, q) is infinitesimally rigid on \mathcal{Y}^q . This follows from the facts that $G_1 - v_2v_3$ is rigid and that $(G - v_0v_1, q)$ can be constructed from $(G_1 - v_2v_3, p_1)$ by adding v_0 and v_1 as vertices of degree two at points which do not lie on the lines joining their two neighbours v_2, v_3 .

Now suppose (ω', λ') is an equilibrium stress for (G_1, p_1) with a maximum rank stress matrix $\Omega_{\text{cyl}}(\omega', \lambda')$. Since $G_1 - v_2v_3$ is rigid, we may suppose (ω', λ') has been chosen so that the stress value on v_2v_3 is non-zero and hence we may scale (ω', λ') so that $\omega'_{01} = -1$. We may combine (ω', λ') with the unique equilibrium stress for $(H, q|_H)$ which has stress value one on v_2v_3 to obtain the equilibrium stress (ω, λ) for (G, q) on \mathcal{Y}^q defined by $\omega_f = \omega'_f$ for all $f \in E_1 - v_2v_3$, $\lambda_v = \lambda'_v$ for all $v \in V_1 - \{v_2, v_3\}$, $\omega_{12} = -\frac{z_3}{z_3-1}$, $\omega_{02} = \frac{x_3}{y_3(z_3x_3-x_3-z_3+1)}$, $\omega_{13} = -z_3 + z_3x_3$, $\omega_{03} = -x_3$, $\omega_{01} = \frac{x_3z_3}{z_3-1}$, $\lambda_{v_2} = \lambda'_{v_2} + \frac{y_3-1}{y_3(z_3-1)}$, $\lambda_{v_3} = \lambda'_{v_3} - 1 + y_3 + x_3 - x_3y_3$, $\lambda_{v_1} = \frac{z_3(1-y_3-x_3+x_3y_3)}{z_3-1}$, and $\lambda_{v_0} = -\frac{x_3(y_3-1)}{y_3(z_3-1)}$.

We have $\Omega(\omega) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where

$$A = \begin{bmatrix} \omega_{13} + \omega_{12} + \omega_{01} & -\omega_{01} \\ -\omega_{01} & \omega_{03} + \omega_{02} + \omega_{01} \end{bmatrix}, \quad B = \begin{bmatrix} -\omega_{02} & -\omega_{03} & 0 & \dots \\ -\omega_{12} & -\omega_{13} & 0 & \dots \end{bmatrix}, \quad C = B^T,$$

and

$$D = \begin{bmatrix} \sum_{j \geq 4} \omega_{2j} + \omega_{02} + \omega_{12} & 0 & -\omega_{24} & \dots \\ 0 & \sum_{j \geq 4} \omega_{3j} + \omega_{03} + \omega_{13} & -\omega_{34} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We can now substitute the values for $\omega_{01}, \omega_{02}, \omega_{03}, \omega_{12}, \omega_{13}$ into A, B, C . This gives

$$A = \begin{bmatrix} -z_3 + z_3x_3 - \frac{z_3}{z_3-1} + \frac{x_3z_3}{z_3-1} & -\frac{x_3z_3}{z_3-1} \\ -\frac{x_3z_3}{z_3-1} & -x_3 + \frac{x_3}{y_3(1-x_3-z_3+x_3z_3)} + \frac{x_3z_3}{z_3-1} \end{bmatrix}.$$

Since $\{x_3, y_3, z_3\}$ is an algebraically independent set, A is invertible and a calculation gives

$$CA^{-1}B = \begin{bmatrix} -\frac{x_3y_3-x_3-y_3}{y_3(z_3x_3-x_3-z_3+1)} & -1 & 0 & \dots \\ -1 & (z_3-1)(x_3-1) & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We may now deduce that

$$D - CA^{-1}B = \begin{bmatrix} \sum_{j \geq 4} \omega_{2j} - 1 & 1 & -\omega_{24} & \dots \\ 1 & \sum_{j \geq 4} \omega_{3j} - 1 & -\omega_{34} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \Omega(\omega')$$

Equation (4.1) now gives $\text{rank } \Omega(\omega) = \text{rank } A + \text{rank } \Omega(\omega') = |V(G)| - 2$.

We may use a similar calculation to deduce that $\text{rank}[\Omega(\omega) + \Lambda(\lambda)] = |V(G)| - 2$. Hence $\text{rank } \Omega_{\text{cyl}}(\omega, \lambda) = 3|V(G)| - 6$. The result now follows from Lemma 4.1. \square

We chose the values of $q(v_0)$ and $q(v_1)$ in the above proof so that the framework $(H, q|_H)$ would have a nowhere zero equilibrium stress. We could also have accomplished this by putting $q(v_0)$ and $q(v_1)$ in the same plane as $q(v_2)$ and $q(v_3)$, but such a choice would have resulted in the matrix A being singular.

We next consider the generalised vertex splitting operation. In [10, Theorem 5.2] it was proved that the standard vertex splitting operation, with the additional assumption that the new graph is rigid when we delete the *bridging edge* (i.e. the edge joining the two copies of the split vertex), preserves generic global rigidity on families of concentric cylinders. We will show that the generalised vertex splitting operation preserves the stronger property of having a maximum rank stress matrix under the same assumption that the bridging edge is redundant.

We will need the following result about frameworks with two coincident points. Let G be a graph with two distinguished vertices u and v . A framework (G, p) is *uv-coincident* if $p(u) = p(v)$. A *uv-coincident* framework is *uv-generic* if $(G - u, p')$ is generic, where p' is the restriction of p to $G - u$.

Theorem 4.3. [10, Theorem 18] *Let u and v be distinct vertices of a graph G and let (G, p) be a *uv-generic*, *uv-coincident* realisation of G on a family of concentric cylinders \mathcal{Y} . Then (G, p) is infinitesimally rigid if and only if the graphs $G - uv$ and G/uv are both rigid on the cylinder.*

Theorem 4.4. *Suppose that (G, p) is an infinitesimally rigid generic framework on a family of concentric cylinders \mathcal{Y} and that (ω, λ) is an equilibrium stress for (G, p) with a maximum rank stress matrix. Let \hat{G} be obtained from G by a generalised vertex splitting operation and suppose that \hat{G} is rigid on the cylinder and has a redundant bridging edge. Then there exists a realisation (\hat{G}, q) in \mathbb{R}^3 which is infinitesimally rigid and has a maximum rank stress matrix on \mathcal{Y}^q .*

Proof. Suppose that \hat{G} is obtained from G by choosing a vertex v_1 with neighbours v_2, v_3, \dots, v_t , deleting the edges $v_1v_2, v_1v_3, \dots, v_1v_k$ and then adding a new vertex v_0 and new edges $v_0v_1, v_0v_2, \dots, v_0v_k$ and v_0x for some vertex x distinct from v_0, v_1, \dots, v_k . Choose an infinitesimally rigid realisation (\hat{G}, q) of \hat{G} such that $q|_G = p$ and $q(v_0) - q(v_1)$ is linearly independent from

$$\{\bar{p}(v_1), \sum_{j=2}^k \omega_{1j}(p(v_1) - p(v_j))\}$$

and from

$$\{\bar{p}(v_1), \sum_{j=k+1}^t \omega_{1j}(p(v_1) - p(v_j))\}$$

where $\bar{p}(v_1)$ is the projection of $p(v_1)$ onto the x, y -plane.

Let (\hat{G}, \hat{p}) be the v_0v_1 -coincident v_0v_1 -generic framework with $\hat{p}|_G = p$ and $\hat{p}(v_0) = p(v_1)$. Then (\hat{G}, \hat{p}) is infinitesimally rigid by Theorem 4.3 (since $\hat{G} - v_0v_1$ and $\hat{G}/v_0v_1 = G + v_0x$ are both rigid on the cylinder). Hence $\text{rank } R_{\text{cyl}}(\hat{G}, \hat{p}) = 3|V(\hat{G})| - 2$. Let $d_{01} = q(v_0) - q(v_1)$ and let $\hat{R}(\hat{G}, \hat{p})$ be obtained from $R_{\text{cyl}}(\hat{G}, \hat{p})$ by replacing the zero row indexed by v_0v_1 by the row with d_{01} and $-d_{01}$ in the v_0 and v_1 columns and zeros elsewhere. The choice of d_{01} tells us there exist unique scalars $\bar{\omega}_{01}, \bar{\omega}_{01}, \bar{\lambda}_1, \bar{\lambda}_1$ such that

$$\sum_{j=2}^k \omega_{ij}(p(v_1) - p(v_j)) + \bar{\lambda}_1 \bar{p}(v_1) + \bar{\omega}_{01} d_{01} = 0$$

and

$$\sum_{j=k+1}^t \omega_{ij}(p(v_1) - p(v_j)) + \bar{\lambda}_1 \bar{p}(v_1) + \bar{\omega}_{01} d_{01} = 0.$$

Since (ω, λ) is an equilibrium stress for (G, p) we have

$$\sum_{j=2}^t \omega_{ij}(p(v_1) - p(v_j)) + \lambda_1 \bar{p}_1 = 0.$$

Hence $(\bar{\lambda}_1 + \bar{\lambda}_1) \bar{p}(v_1) + (\bar{\omega}_{01} + \bar{\omega}_{01}) d_{01} = 0$. It follows that $\lambda_1 = \bar{\lambda}_1 + \bar{\lambda}_1$ and $\bar{\omega}_{01} = -\bar{\omega}_{01}$.

Let $\hat{\omega}_{ij} = \omega_{ij}$ if $(i, j) \neq (0, 1)$, $\hat{\omega}_{01} = \bar{\omega}_{01}$, $\hat{\omega}_{0x} = 0$, $\hat{\lambda}_i = \lambda_i$ if $i \neq 0, 1$, $\hat{\lambda}_0 = \bar{\lambda}_1$ and $\hat{\lambda}_1 = \bar{\lambda}_1$. Then $(\hat{\omega}, \hat{\lambda}) \in \text{coker } \hat{R}(\hat{G}, \hat{p})$. On the other hand $(\hat{G} - v_0 v_1, \hat{p})$ is infinitesimally rigid and $\hat{R}(\hat{G} - v_0 v_1, \hat{p}) = R(\hat{G} - v_0 v_1, \hat{p})$ so $\text{rank } \hat{R}(\hat{G} - v_0 v_1, \hat{p}) = 3|V(\hat{G})| - 2$. This implies that the rows of $\hat{R}(\hat{G} - v_0 v_1, \hat{p})$ are linearly independent. Since $(\hat{\omega}, \hat{\lambda})$ is a non-zero vector in $\text{coker } \hat{R}(\hat{G}, \hat{p})$ we must have $\hat{\omega}_{01} \neq 0$.

Now consider the effect of replacing d_{01} by cd_{01} for some $c \in \mathbb{R} \setminus \{0\}$. This will replace $(\hat{\omega}, \hat{\lambda})$ by (ω^c, λ^c) where $\omega_{01}^c = \frac{1}{c}\hat{\omega}_{01}$, $\omega_{ij}^c = \hat{\omega}_{ij}$ for all $(i, j) \neq (0, 1)$ and $\lambda_i^c = \hat{\lambda}_i$ for all i . The matrix $\Omega(\omega^c)$ defined by (ω^c, λ^c) has the form

$$\Omega(\omega^c) = \begin{bmatrix} \frac{1}{c}\omega_{01}^1 + \sum_{j \geq 2} \omega_{0j}^1 & -\frac{1}{c}\omega_{01}^1 & -\omega_{02}^1 & \dots \\ -\frac{1}{c}\omega_{01}^1 & \frac{1}{c}\omega_{01}^1 + \sum_{j \geq 2} \omega_{1j}^1 & -\omega_{12}^1 & \dots \\ -\omega_{02}^1 & -\omega_{12}^1 & \sum_{j \geq 2} \omega_{2j}^1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

When we add row 1 to row 2 then column 1 to column 2 and use the facts that $\omega_{1j}^1 + \omega_{0j}^1 = \omega_{1j}$ for all $2 \leq j \leq k$, we obtain

$$\begin{bmatrix} \frac{1}{c}\omega_{01}^1 + \sum_{j \geq 2} \omega_{0j}^1 & \sum_{j \geq 2} \omega_{0j}^1 & -\omega_{02}^1 & \dots \\ \sum_{j \geq 2} \omega_{0j}^1 & \sum_{j \geq 2} \omega_{1j}^1 & -\omega_{12}^1 & \dots \\ -\omega_{02}^1 & -\omega_{12}^1 & \sum_{j \geq 2} \omega_{2j}^1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{c}\omega_{01}^1 + \sum_{j \geq 2} \omega_{0j}^1 & \sum_{j \geq 2} \omega_{0j}^1 & -\omega_{02}^1 & \dots \\ \sum_{j \geq 2} \omega_{0j}^1 & & & \\ -\omega_{02}^1 & & \Omega(\omega) & \\ \vdots & & & \end{bmatrix}$$

We can now use Equation (4.1) with $D = \Omega(\omega)$ to deduce that

$$\text{rank } \Omega(\omega^c) = \text{rank}(a) + \text{rank}(\Omega(\omega) - a^{-1}B^T B)$$

where $a = \frac{1}{c}\omega_{01}^1 + \sum_{j \geq 2} \omega_{0j}^1$ and $B = (\sum_{j \geq 2} \omega_{0j}^1, -\omega_{02}^1, \dots, -\omega_{0n}^1)$. Since $\omega_{01}^1 = \hat{\omega}_{01} \neq 0$, we have $\lim_{c \rightarrow 0} a = \pm\infty$ and hence $\text{rank } \Omega(\omega^c) = 1 + \text{rank } \Omega(\omega)$ when c is sufficiently close to zero.

A similar argument can be used to deduce that

$$\text{rank}(\Omega(\omega^c) + \Lambda(\lambda^c)) = 1 + \text{rank}(\Omega(\omega) + \Lambda(\lambda)).$$

Hence $\text{rank } \Omega_{\text{cyl}}(\omega^c, \lambda^c) = \text{rank } \Omega_{\text{cyl}}(\omega, \lambda) + 3 = 3|V(\hat{G})| - 6$ for all c sufficiently close to zero.

For each such $c \in \mathbb{R} \setminus \{0\}$, choose $q_c \in \mathbb{R}^{|V(\hat{G})|}$ such that $q_c|_G = p$ and $q_c(v_0) - p(v_1) = c d_{01}$. By making c sufficiently close to zero, we can choose an $(\hat{\omega}^c, \hat{\lambda}^c) \in \text{coker } R_{\text{cyl}}(\hat{G}, q_c)$ such that $(\hat{\omega}^c, \hat{\lambda}^c)$ is arbitrarily close to (ω^c, λ^c) . Then $(\hat{\omega}^c, \hat{\lambda}^c)$ will be a stress for (\hat{G}, q_c) on \mathcal{Y}^{q_c} with $\text{rank } \Omega_{\text{cyl}}(\hat{\omega}^c, \hat{\lambda}^c) = 3(|V| + 1) - 6$ for all c sufficiently close to zero. \square

5. GLOBALLY RIGID FRAMEWORKS

We start this section by proving Theorem 1.3.

Proof of Theorem 1.3. We apply induction on $|V|$. We gave specific infinitesimally rigid realisations of $K_5 - e$ and H_1 on a family of concentric cylinders which have equilibrium stresses with maximum rank stress matrices in [12]. Theorem 1.1 and Lemma 4.1 now imply that every generic realisation p of $K_5 - e$ or H_1 in \mathbb{R}^3 is infinitesimally rigid and has a maximum rank stress matrix on \mathcal{Y}^p . By Theorem 2.1 any circuit G can be formed from

$K_5 - e$ or H_1 by K_4^- -extensions and generalised vertex splits. The result now follows from Theorems 1.6, 4.2 and 4.4. \square

We will need some further concepts and results from matroid theory to deduce Theorem 1.2 from Theorem 1.3. A matroid is *connected* if any pair of edges is contained in a common circuit.

Lemma 5.1. [16, Theorem 5.4] *Suppose that G is a graph. Then $\mathcal{M}_{2,2}^*(G)$ is connected if and only if G is 2-connected and redundantly rigid.*

Let $\mathcal{M} = (E, \mathcal{J})$ be a matroid and let C_1, C_2, \dots, C_t be a non-empty sequence of circuits of \mathcal{M} . Let $D_j = C_1 \cup C_2 \cup \dots \cup C_j$ for $1 \leq j \leq t$ and suppose $D_t = E$. We say that C_1, C_2, \dots, C_t is an *ear decomposition* of \mathcal{M} if for all $2 \leq i \leq t$ the following properties hold:

- (1) $C_i \cap D_{i-1} \neq \emptyset$;
- (2) $C_i - D_{i-1} \neq \emptyset$;
- (3) no circuit C'_i satisfying (1) and (2) has $C'_i - D_{i-1}$ properly contained in $C_i - D_{i-1}$.

Lemma 5.2 ([6]). *A matroid is connected if and only if it has an ear decomposition.*

We also need a ‘glueing’ lemma for combining globally rigid frameworks. Its proof uses the following result.

Lemma 5.3. [11, Lemma 14] *Let G be a graph with at least five vertices and (G, p) and (G, q) be congruent generic realisations of G on a family of concentric cylinders \mathcal{Y} . Then $\iota \circ p = q$ for some isometry ι of \mathcal{Y} .*

Note that the isometries of any family of concentric cylinders \mathcal{Y} are translations along and rotations about the z -axis, as well as reflections in any plane containing or orthogonal to the z -axis. Hence Lemma 5.3 implies that, if (G, p) and (G, q) are congruent generic framework on \mathcal{Y} with at least five vertices satisfying $p(v_1) = q(v_1)$ and $p(v_2) = q(v_2)$ for two distinct vertices v_1, v_2 of G , then $p = q$.

Lemma 5.4. *Let G_1 and G_2 be graphs on at least five vertices and with at least two vertices in common. Let (G, p) be a generic realisation of $G = G_1 \cup G_2$ in \mathbb{R}^3 and let $p_i = p|_{G_i}$. Suppose that (G_i, p_i) is globally rigid on \mathcal{Y}^{p_i} for $i = 1, 2$. Then (G, p) is globally rigid on \mathcal{Y}^p .*

Proof. Choose $u, v \in V(G_1) \cap V(G_2)$ and let (G, q) be an equivalent framework to (G, p) on \mathcal{Y}^p . By applying a suitable isometry of \mathcal{Y}^p to q we may assume that $p(u) = q(u)$. Since (G_1, p_1) is globally rigid on \mathcal{Y}^{p_1} , Lemma 5.4 tells us that $q|_{G_1} = \iota \circ p_1$ for some discrete isometry ι of \mathcal{Y}^{p_1} . In particular $q(v) = (\iota \circ p)(v)$. Since (G_2, p_2) is globally rigid on \mathcal{Y}^{p_2} , Lemma 5.4 implies that there is a unique equivalent realisation of G_2 on \mathcal{Y}^{p_2} which maps u to $p(u)$ and v to $(\iota \circ p)(v)$. Since both $(G_2, q|_{G_2})$ and $(G_2, \iota \circ p_2)$ have this property, $q|_{G_2} = \iota \circ p_2$. Hence $q = \iota \circ p$ and (G, p) is congruent to (G, q) . \square

Proof of Theorem 1.2. Necessity follows from [11]. We prove sufficiency by induction on $|E|$. Since G is 2-connected and redundantly rigid, $\mathcal{M}_{2,2}^*(G)$ is connected by Lemma 5.1, so has an ear decomposition by Lemma 5.2. Let H_1, H_2, \dots, H_t be the circuits induced by an ear decomposition of $\mathcal{M}_{2,2}^*(G)$ and $p_i = p|_{H_i}$. Then (H_i, p_i) is globally rigid on \mathcal{Y}^{p_i} for all $1 \leq i \leq t$ by Theorem 1.3. Since $|V(H_i)| \geq 5$ and $|V(H_i) \cap V(H_{i+1})| \geq 2$ for all $1 \leq i < t$, we may deduce that (G, p) is globally rigid on \mathcal{Y}^p by repeated applications of Lemma 5.4. \square

Theorem 1.2 implies that global rigidity on families of concentric cylinders is a generic property in \mathbb{R}^3 . That is, if G is a graph and p is generic in \mathbb{R}^3 , then (G, p) is globally rigid on \mathcal{Y}^p if and only if (G, q) is globally rigid on \mathcal{Y}^q for all generic $q \in \mathbb{R}^3$. It also gives a polynomial

time algorithm for checking generic global rigidity on families of concentric cylinders since 2-connectivity [20] and redundant rigidity [3, 14] can both be checked efficiently.

6. CLOSING REMARKS

1) We conjecture that the combinatorial characterisation of global rigidity given in Theorem 1.2 remains true for frameworks realised generically on the unit cylinder, or more generally any family of concentric spheres. To extend the proof technique of this paper would first require improving Theorem 1.6, see [12, Conjecture 1] and the discussion in [12, Section 8]. Note that the characterisation of global rigidity in the plane given in [9] and the correspondence between the plane and the unit sphere given in [5, Theorem 12] imply that a generic framework (G, p) on the unit sphere is globally rigid if and only if G is 3-connected and redundantly rigid on the sphere. Moreover, the proof technique of [5, Theorem 12] can be used to show that the same result holds for a generic framework on any family of concentric spheres.

2) We also conjecture that the converse to Theorem 1.6 holds. That is, a generic framework (G, p) in \mathbb{R}^3 which is globally rigid on \mathcal{Y}^p must have a maximum rank stress matrix. This would be an analogue of the main result of [7]. It follows from our work that this conjecture holds when G is a circuit in $\mathcal{M}_{2,2}^*$ since we have shown that every circuit is both globally rigid and has a maximum rank stress matrix. One way to extend this to all graphs would be to prove a version of Lemma 5.4 showing that the ‘glueing’ operation preserves the property of having a maximum rank stress matrix. We can adapt the proof of Connelly [4, Lemma 10] to prove the special case when G_1 and G_2 have exactly 2 vertices in common but the general case seems challenging. An alternative approach would be to adapt the proof technique given in [7] to generic frameworks on families of concentric cylinders.

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